Final Examination, Differential Equations, 2021. B. Math 3rd year

Total points– Maximum 60, Date– 17th May 2021.

1. (3+3+4 points)

   (I) Show that following equation is an exact equation and then solve–

   $$(y - x^3)dx + (x + y^3)dy = 0.$$

   (II) Solve the following as linear equation–

   $$(2y - x^3)dx = xdy.$$

   (III) Show that $f(x, y) = xy^2$

   (i) satisfies a Lipschitz condition on any rectangle $a \leq x \leq b$ and $c \leq y \leq d$;

   (ii) does not satisfy a Lipschitz condition on any strip $a \leq x \leq b$ and $-\infty < y < \infty$.

2. (5 points) Show that $y = c_1 x + c_2 x^2$ is the general solution of

   $$x^2 y'' - 2xy' + 2y = 0$$

   on any interval not containing 0, and find the particular solution for which $y(1) = 3$ and $y'(1) = 5$.

3. (5+5 points) Derive the normal form of Bessel's equation

   $$x^2 y'' + xy' + (x^2 - p^2)y = 0,$$

   and use it to show that every nontrivial solution has infinitely many positive zeros.

4. (5+5+5 points) Consider the equation

   $$y'' + xy' + y = 0.$$

   (I) Find its general solution $y = \sum a_n x^n$ in the form $y = a_0 y_1(x) + a_1 y_2(x)$, where $y_1(x)$ and $y_2(x)$ are power series.

   (II) Use the ratio test to verify that the two series $y_1(x)$ and $y_2(x)$ converge for all x.

   (III) Show that $y_1(x)$ is the series expansion of $e^{-x^2/2}$, and find a second independent solution by using the known solution $y_1(x)$.

5. (10 points) Consider the following Bessel's equation for $p = 0$

   $$x^2 y'' + xy' + x^2 y = 0.$$

   Show that its indicial equation has only one root and deduce that

   $$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

   is the corresponding Frobenius series solution. Also show that the series converges for all x.

6. (3+3+4 points)

   (I) Verify by examining the series expansions of the function on l.h.s.
   $$\log(1+x) = xF(1,1,2,-x).$$

   (II) Validate the following statement without attempting to justify the limit processes involved
   $$\cos x = \lim_{a\to\infty} F\left(a, a, \frac{1}{2}, \frac{-x^2}{4a^2}\right).$$

   (III) Find the general solution of the following differential equation near the singular point, $x = 0$
   $$x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0.$$

7. (10 points) Let
   $$D = \{x, y \,:\, x = r\cos\theta, y = r\sin\theta,\ 0 < r_0 < r < r_1,\ 0 \le \theta < \pi\}.$$
   Solve the Dirichlet problem, for $\Delta u = 0$ in D, $u(r_1,\theta) = f_1(\theta)$ and $u(r_0,\theta) = f_0(\theta)$, $0 \le \theta < \pi$.

   [**Hint:** You may assume a convenient Fourier series expansion for $f_1(\theta)$ and $f_0(\theta)$.]